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Poincaré Canonical Momenta and Nambu Mechanics

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Abstract

We show that if certain Poincaré-like integrals are conserved, then to each configuration coordinate of a system an entity can be associated that is an acceptable generalization of the notion of canonical momentum: In the particular case of standard mechanics, the canonical momenta are retrieved. Under certain general restrictions, the Poincaré momenta make sense for either mechanical or general systems for which we do not have (or are not aware of) entities (like the Lagrangian) that are generally used to define the momentum. The Poincaré momentum may also make sense for systems whose characteristics are difficult, or impossible, to reconcile with the notion of the usual canonical momentum. It is also relevant for certain cases where a Lagrangian exists, but it leads to a mixture of physical and unphysical entities. In particular, we show that while physical canonical momenta do not generally exist in the new Nambu mechanics (because of the dimensionality of state vector space), the Poincaré momenta exist, they are physical, and have the properties we could have expected for the mechanics.

1. *Statement of the Problem*

The phase space of standard analytical mechanics has even dimension. If physical systems exist whose phase-space dimension is odd, they cannot be properly formulated (or perhaps even conceived) without the extension of mechanics to odd dimension. Nambu (1973) showed a possible way of doing this extension by creating a new mechanics (referred to in what follows as Nambu mechanics) whose principles and main properties allowed for odd as well as even dimension (see also Ruggeri, 1975; Cohen and Kálnay 1975; García-Sucre and Kálnay, 1975; Kálnay, 1974). Bunge[1] has pointed out that because the Nambu formalism does not necessarily refer to mechanical systems, it therefore is a theory of general systems even though formulated in the language of mechanics. We shall use the term *Nambu theory* for the

[1] M. Bunge, private communication.

Nambu theory of general systems and *Nambu mechanics* for purely mechanical systems.

In the Nambu theory it is not a priori clear which are the canonical momenta conjugated to the coordinates. Notice that the standard notion of canonical momenta

$$p^{\alpha} = \partial L/\partial \dot{q}_{\alpha} \tag{1.1}$$

forbids odd-dimensional phase space.[2] The purpose of this paper is to try to locate what canonical momenta are in Nambu theory. There existence is also studied. Our procedure is the following: We shall associate to each coordinate (or set of coordinates) in Nambu theory a dynamical variable (or a set of dynamical variables), which we shall call the Poincaré momenta conjugated to the former coordinates. (We use the word "Poincaré" as a reminder that we obtain the new definition from Poincaré-like invariant integrals.) We then prove that in the particular case of standard Hamiltonian dynamics the Poincaré momentum conjugated to a given configuration-space coordinate coincides with the standard canonical momentum. The new Poincaré momenta can then be properly considered as a generalization of the standard momenta. Finally we shall show that the Poincaré momenta make sense in Nambu theory and, moreover, that there consideration as momenta is consistent with other properties of the Nambu formalism.

The Nambu theory for one multiplet $(x_1, x_2, \ldots, x_n)$ corresponds to the Hamilton formalism for a system with only one configuration variable q and its canonically conjugated momentum p (see, e.g., Cohen and Kálnay, 1975, Section 1). In order to have the Nambu analog of Hamiltonian systems with an arbitrary number of configuration-space coordinates, Nambu systems with an arbitrary number of multiplets must be considered in the sense of equation (6) of Nambu (1973). In order to fix the notation, we shall write the main formulas of Nambu systems with an arbitrary number of multiplets in such a form that the comparison with the Hamilton formalism becomes transparent. Let n be the dimensionality and μ the number of the Nambu multiplets, $\mu = 1, 2, \ldots$. We call

$$x_{\alpha}, \qquad \alpha = 1, 2, \ldots, \mu \tag{1.2a}$$

any of the Nambu multiplets

$$x_{\alpha} \stackrel{\text{df}}{=} (x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha i}, \ldots, x_{\alpha n}), \qquad i = 1, 2, \ldots, n \tag{1.2b}$$

In the Nambu formalism [in the sense of equation (6) of Nambu (1973)] there are $n - 1$ Hamiltonians

$$H_1, H_2, \ldots, H_{n-1} \tag{1.3}$$

[2] In spite of this the coordinates $x_{\alpha i}$ of Nambu theory are correctly called phase-space coordinates because all of them span the state vector space of the c-number form of the Nambu theory. Further discussion is given in Section 4.1.

and given a dynamical variable $F(x_1, x_2, \ldots, x_\alpha, \ldots, x_\mu)$ the Nambu equation of motion is

$$\dot{F} = \sum_{\alpha=1}^{\mu} \frac{\partial(F, H_1, \ldots, H_{n-1})}{\partial(x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha n})} \tag{1.4}$$

where $\partial(\ldots)/\partial(\ldots)$ is a Jacobian. By introducing the *Nambu bracket*[3] (or generalized Poisson brackets in the terminology of Nambu, 1973)

$$\{F_1, F_2, \ldots, F_n\} \stackrel{\text{df}}{=} \sum_{\alpha=1}^{\mu} \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha n})} \tag{1.5}$$

among the dynamical variables $F_1, F_2, \ldots, F_n$ equation (1.4) can be rewritten as

$$\dot{F} = \{F, H_1, \ldots, H_{n-1}\} \tag{1.6}$$

The phase-space coordinates in Nambu theory are the $x_{\alpha r}$; they are such that

$$\{x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha n}\} = 1, \qquad \alpha = 1, 2, \ldots, \mu \tag{1.7a}$$

$$\left.\begin{array}{l}\{x_{\alpha_1 i_1}, x_{\alpha_2 i_2}, \ldots, x_{\alpha_n i_n}\} = 0 \qquad \text{if at least} \\ \text{two of the } \alpha_r \text{ are different and/or if at least} \\ \text{two of the } i_r \text{ are equal}\end{array}\right\} \tag{1.7b}$$

Let us now consider a standard Hamiltonian system whose configuration-space coordinates are

$$q_1, q_2, \ldots, q_\alpha, \ldots, q_\mu \tag{1.8a}$$

and let

$$p^1, p^2, \ldots, p^\alpha, \ldots, p^\mu \tag{1.8b}$$

be the corresponding canonically conjugated momenta. The single Hamiltonian H generates the time evolution according to

$$\dot{F}(q,p) = \sum_{\alpha=1}^{\mu} \left[\frac{\partial F(q,p)}{\partial q_\alpha} \frac{\partial H(q,p)}{\partial p^\alpha} - \frac{\partial F(q,p)}{\partial p^\alpha} \frac{\partial H(q,p)}{\partial q_\alpha}\right] \tag{1.9}$$

The Poisson bracket is

$$\{F_1, F_2\} \stackrel{\text{df}}{=} \sum_{\alpha=1}^{\mu} \frac{\partial(F_1, F_2)}{\partial(q_\alpha, p^\alpha)} \tag{1.10}$$

[3] These brackets (also called Poisson brackets of the nth order) were previously used in a different context. See the result by Albeggiani in Example 5 (p. 337) of paragraph 153 (Whittaker, 1937). We are indebted to F. Marín (private communication) for letting us know of Albeggiani's result.

so that for any dynamical variable $F(q, p)$

$$\dot{F} = \{F, H\} \tag{1.11}$$

The phase-space coordinates q_α, p^α are such that

$$\{q_\alpha, p^\alpha\} = 1 \tag{1.12a}$$

$$\begin{aligned} &\{q_{\alpha_1}, p^{\alpha_2}\} = 0 \qquad \text{if } \alpha_1 \neq \alpha_2 \\ &\{q_{\alpha_1}, q_{\alpha_2}\} = \{p^{\alpha_1}, p^{\alpha_2}\} = 0, \qquad \forall \alpha_1, \alpha_2 \end{aligned} \tag{1.12b}$$

[The complicated form of stating equations (1.12b) was used to make easier the comparison with equation (1.7b).] If we call in the Hamiltonian formalism

$$x_{\alpha 1} \overset{\text{df}}{=} q_\alpha, \qquad x_{\alpha 2} \overset{\text{df}}{=} p^\alpha, \qquad x_\alpha \overset{\text{df}}{=} (q_\alpha, p^\alpha), \qquad \alpha = 1, 2, \ldots, \mu \tag{1.13}$$

then the Hamiltonian dynamics coincide with the Nambu dynamics for a Nambu system of doublets ($n = 2$). In fact, equations (1.8)–(1.13), respectively, go to equations (1.2) and (1.4)–(1.7). Moreover, the Nambu bracket is totally antisymmetric, as is the Poisson bracket. Notice that equation (1.12a) corresponds to (1.7a) and (1.12b) to (1.7b), etc.

The Nambu formalism is thus a proper generalization of the Hamilton formalism.

2. *Poincaré Momenta, General Case*

Here we shall consider a general theory of systems, in particular a general mechanics, without restriction to any particular case like the Nambu or Hamilton ones. Our only assumptions will be the following: (1) To consider c-number systems. (ii) That a linear state vector space (which we call $\mathscr{E}$) exists. (It is not essential that it be finite dimensional, but we shall consider it to be so for definiteness.) Call S the dimension. (iii) That integral constants of motion like that of equation (2.11) (see below) exist.

Let μ and n be two positive integers such that

$$\mu n = S \tag{2.1}$$

The entities to be defined below depend on the selection of μ and n, so that from now on we consider the selected values μ and n as fixed. There are S coordinates in $\mathscr{E}$, which because of equation (2.1) can be labeled with two indices. We write them as before

$$x_{\alpha i}, \qquad \alpha = 1, 2, \ldots, \mu, \qquad i = 1, 2, \ldots, n \tag{2.2}$$

Definition 2.1. An n-plet is any of the sets

$$x_\alpha \overset{\text{df}}{=} (x_{\alpha 1}, x_{\alpha 2}, \ldots, x_{\alpha i}, \ldots, x_{\alpha n}), \qquad \alpha = 1, 2, \ldots, \mu \tag{2.3}$$

μ being the number of the n-plets. We call x any vector of $\mathscr{E}$. Then we have

$$x = (x_1, x_2, \ldots, x_\alpha, \ldots, x_\mu). \tag{2.4}$$

Definition 2.2. Let s be an integer such that

$$1 \leqslant s \leqslant n \tag{2.5}$$

Let us consider a set

$$q_\alpha \overset{\text{df}}{=} (q_{\alpha 1}, q_{\alpha 2}, \ldots, q_{\alpha r}, \ldots, q_{\alpha s}) \tag{2.6}$$

of a complex (or real) valued independent functions

$$\begin{gathered} q_{\alpha r}: x \to q_{\alpha r}(x) \in \mathbb{C} \quad (\text{or } \mathbb{R}) \\ r = 1, 2, \ldots, s, \qquad \alpha = 1, 2, \ldots, \mu \end{gathered} \tag{2.7a}$$

We also denote

$$q \overset{\text{df}}{=} (q_1, q_2, \ldots, q_\alpha, \ldots, q_\mu) \tag{2.7b}$$

An s coordinate of configuration space is any of the sets q_α.

Definition 2.3. A configuration space coordinate $q_\alpha \equiv q_{\alpha 1}$ is a 1-coordinate.

Definition 2.4. Let us consider (if it exists) a set

$$f^\alpha \overset{\text{df}}{=} (f_1^\alpha, f_2^\alpha, \ldots, f_v^\alpha, \ldots, f_{n-s}^\alpha) \tag{2.8}$$

of $n - s$ complex (or, respectively, real) valued independent functions

$$\begin{gathered} f_v^\alpha: x \to f_v^\alpha(x) \in \mathbb{C} \quad (\text{or } \mathbb{R}) \\ v = 1, 2, \ldots, n - s, \qquad \alpha = 1, 2, \ldots, \mu \end{gathered} \tag{2.9}$$

such that, for a fixed s and for any α there would be

$$\frac{dJ_n^s(\sigma)}{dt} = 0 \tag{2.10}$$

where σ is an arbitrary n-dimensional surface belonging to $\mathscr{E}$ and where

$$J_n^s(\sigma) \overset{\text{df}}{=} \sum_{\alpha=1}^{\mu} \int_\sigma dq_{\alpha 1} \cdots dq_{\alpha r} \cdots dq_{\alpha s} df_1^\alpha \cdots df_v^\alpha \cdots df_{n-s}^\alpha \tag{2.11}$$

We also denote

$$f \overset{\text{df}}{=} (f^1, f^2, \ldots, f^\alpha, \ldots, f^\mu) \tag{2.12}$$

Then, *whenever f exists, an $(n - s)$-Poincaré momentum conjugated to the s coordinate q_α is the set f^α.*

Definition 2.5. A Poincaré momentum $f^\alpha \equiv f_1^\alpha$ canonically conjugated to the coordinate q_α is an 1-Poincaré momentum conjugated to q_α.

3. *The Poincaré Momentum is the Canonical Momentum for Standard Systems*

Let us consider a standard Hamiltonian system so that equations (1.8)-(1.12) hold; or, let us start from a standard Lagrangian system so that the usual canonical momenta (always denoted p^α) are defined by equation (1.1). We shall prove that for such systems the Poincaré momentum (always denoted f^α) coincides with p^α. We shall work within the Hamilton formalism. Because of equation (1.8) we have that in equation (2.1) $S = 2\mu$, so that

$$n = 2 \tag{3.1}$$

The 2-plets (Definition 2.1) can always be taken as

$$x_\alpha = (q_\alpha, p^\alpha), \qquad \alpha = 1, 2, \ldots, \mu \tag{3.2}$$

Because of Definition 2.3 the standard configuration-space coordinates coincide with the 1-coordinates, so that the usual configuration-space coordinates of the Hamiltonian formalism coincide with the configuration-space coordinates q_α introduced in Definition 2.3. Then, in order to compare the formalism of Section 2 with the standard one we must select

$$s = 1 \tag{3.3}$$

in equation (2.5). Notice that because of equation (3.2) we have

$$f^\alpha(x) = f^\alpha(q, p) \tag{3.4}$$

Finally, we remark that we call H the usual Hamiltonian, which satisfies equation (1.9).

Theorem 3.1. The canonical momenta p^α are also Poincaré momenta conjugated to the same q_α.

Proof. Put $f^\alpha = p^\alpha$ in Definition 2.4. Then from the theory of Poincaré-invariant integrals it follows immediately that p^α is an 1-Poincaré momentum conjugated to q_α so that because of Definition 2.5, the proof is ended. □

Theorem 3.2. Let $f^\alpha(q, p)$, $\alpha = 1, 2, \ldots, \mu$ a set of Poincaré momenta respectively conjugated to the q_α. Then $H^*(q, p)$ exists such that

$$\dot{q}_\alpha = \frac{\partial H^*}{\partial f^\alpha} \tag{3.5a}$$

and

$$\dot{f}^\alpha = -\frac{\partial H^*}{\partial q_\alpha} \tag{3.5b}$$

Proof. Being the f^α Poincaré momenta, Definitions 2.4 and 2.5 imply that

$$J_2^1(\sigma) = \sum_{\alpha=1}^{\mu} \iint dq_\alpha \, df^\alpha \tag{3.6}$$

is conserved in time. Because of equations (3.4) and (1.9) the set $(q(t), f(t))$ is the solution of first-order equations. The conservation of the absolute invariant (3.6) implies that of the relative invariant

$$\sum_{\alpha=1}^{\mu} \oint f^{\alpha} dq_{\alpha}$$

(Section 114, p. 271 of Whittaker, 1937), so that a function H^* exists such that equations (3.5) are satisfied (Section 116, p. 274 of Whittaker, 1937). □

Theorem 3.3. The transformation

$$(q_{\alpha}, p^{\alpha}, H) \to (q_{\alpha}, f^{\alpha}, H^*) \tag{3.7}$$

is canonical.[4]

Proof. From equations (3.6) and (2.10) it results that

$$\sum_{\alpha=1}^{\mu} \oint f^{\alpha} dq_{\alpha} \tag{3.8}$$

is a relative integral invariant; therefore as a result of the theorem by Lee Hwa-Chung (1947) (also Gantmacher, 1970, Section 22), the integral (3.8) is proportional to that obtained by replacing in (3.8) f^{α} by p^{α}. Then,

$$\sum_{\alpha=1}^{\mu} \oint (f^{\alpha} - cp^{\alpha}) dq_{\alpha} = 0 \tag{3.9}$$

where c is a constant, so that

$$dF \stackrel{\text{df}}{=} \sum_{\alpha=1}^{\mu} [(f^{\alpha} - cp^{\alpha}) dq_{\alpha} + 0 \cdot dp^{\alpha}] \tag{3.10}$$

is an exact differential. This implies that F exists such that

$$\frac{\partial F}{\partial p^{\alpha}} = 0 \tag{3.11a}$$

and

$$f^{\alpha}(q, p) = cp^{\alpha} + \frac{\partial F(q)}{\partial q_{\alpha}} \tag{3.11b}$$

Now we can compute the Poisson brackets:

$$\{q_{\alpha}, q_{\beta}\} = 0 \tag{3.12a}$$

$$\{q_{\alpha}, f^{\beta}\} = c\delta_{\alpha}{}^{\beta} \tag{3.12b}$$

$$\{f^{\alpha}, f^{\beta}\} = 0, \qquad \forall \alpha, \beta \tag{3.12c}$$

[4] As regards canonicity, the terminology is not uniform. We adhere to that of Currie and Saletan (1972) and Saletan and Cromer (1971).

This is a necessary and sufficient condition for (3.7) to be canonical (Currie and Saletan, 1972; Saletan and Cromer, 1971, Section 6.3; Gantmacher, 1970, Section 32; Sudarshan and Mukunda, 1974, Chap. V). We stress that the nomenclature we use is that of Currie, Saletan, and Cromer. □

Note 3.4. It is not necessary for the transformation to be restricted canonically (Saletan and Cromer, 1971), or "univalent canonical" in Gantmacher's language, since $c \neq 1$ is possible, as shown by the example $f^\alpha = cp^\alpha$, $\forall c \neq 0$.

Note 3.5. Canonoid transformations are those that preserve the Hamilton formalism for at least *one* given dynamical system (i.e., for at least the given H), while the canonical transformations are transformations that are canonoid for *any* H consistent with the given phase space. (Currie and Saletan, 1972; Saletan and Cromer, 1971.) It is clear that in order to preserve the physics of a given classical Hamiltonian system, it is enough that the transformation be canonoid; it is even better if it is canonical, but that is not physically essential. With this introduction, we remark that if instead of canonicity we had asked the transformation (3.7) to be canonoid, our result would have been a direct consequence of Theorem 3.2, without having to resort to the involved proof needed to show canonicity.

Note 3.6. The factor c in equations (3.12) can be reabsorbed into f^α resulting in a scale (if $c > 0$) [or reflection and scale (if $c < 0$)] transformation (cf. the end of the proof of Theorem 3.3).

> *Corollary* 3.7. The transformation (3.7) is (up to a scale and/or reflection transformation) a gauge transformation.

Proof. (We use the term gauge transformation in analytical mechanics in the sense of Lévy-Leblond, 1969.) Let us add to the Lagrangian a total time derivative $\dot{F}(q)/c$, where $c \neq 0$ is a constant. Then the canonical momenta change as

$$p^\alpha \to p^\alpha + c^{-1} \frac{\partial F}{\partial q_\alpha}$$

which differs from (3.7) by a scale and/or reflection factor c, as shown by equation (3.12b). □

Remark 3.8. Because of Theorem 3.3, the Poincaré momenta f^α are also canonical momenta for the case of Hamiltonian or Lagrangian systems. (If canonicity is intended in the "restricted" sense, this is true up to a scale and/or reflection transformation of momenta.) Because of this, we can state that *the notion of Poincaré momenta is (when those momenta exist) a valid generalization of that of the standard momenta for those formalisms (such as the Nambu one) for which canonical momenta cannot be introduced in the usual way.* Since the $(n - s)$-Poincaré momenta are a generalization of the Poincaré momenta (Definitions 2.4 and 2.5), *the same can be stated for the $(n - s)$-Poincaré momenta.*

Note 3.9. If p^α is the canonical momentum conjugated to q_α, then it is known that *in the "restricted" canonical sense* $-q_\alpha$ is the momentum conjugated to p^α. On the other hand, if f^α is the Poincaré momentum conjugated to q_α, it results from Definition 2.5 that $+q_\alpha$ is a Poincaré momentum conjugated to f^α. Thus, there is a sign difference; however, there is no contradiction with Remark 3.8 because c can equal -1 (reflection).

Remark 3.10. We began this section restricted to standard Hamiltonian systems. Here the word "standard" is used to exclude generalized classical Hamiltonian mechanics like that of Dirac (1950, 1951, 1958, 1964), for which the results of the present section do not necessarily apply. Let us illustrate the kind of difficulties that can be found with an example concerning Dirac's mechanics, which is the correct one for phase-space-constrained systems. We choose as the example the following: let the Lagrangian be

$$L = \tfrac{1}{2}\dot{q}_1^{\,2} + q_2\dot{q}_2$$

Then q_1 is a linear function of time while q_2 is an arbitrary one. The momenta are $p_1 = dq_1/dt, p_2 = q_2$. Clearly q_1, p_1 are physical variables, and q_2, p_2 are physically irrelevant. The Liouville theorem holds in the physical phase space of coordinates q_1, p_1. However, this is not the case in the conceptually relevant academic *whole* phase space of coordinates q_1, q_2, p_1, p_2:

$$\sum_\alpha \left[\frac{\partial \dot{q}_\alpha}{\partial q_\alpha} + \frac{\partial \dot{p}_\alpha}{\partial p_\alpha}\right]$$

is an undeterminated expression so that Liouville theorem does not hold. In consequence, J_2^1 [equation (3.6)] is not a conserved invariant integral for the whole phase space, although it is for the physical phase space. Therefore, we cannot use Definitions 2.4 and 2.5 to introduce the Poincaré momenta. This drawback of the Poincaré momenta is shared by the canonical momenta too; In fact, p_2 can be formally defined as the generator of translations in the q_2 variable.[5] But, (i) such translations are unphysical because they refer to the unphysical variable q_2 and (ii) they violate the constraint $p_2 = q_2$. If the unphysical variables are dropped altogether, then the system is a standard Hamiltonian. The previous theorems hold, and both the Poincaré and the canonical momentum exist and coincide. Let us stress that we offered this physically uninteresting particular case as an illustration of the difficulties of nonstandard systems. We do not pretend, however, that the behaviour of that system is a typical one for physically relevant systems. We consider that the momentum problem for phase-space constrained systems deserves further consideration.

[5] For canonical (and generalized) transformation in phase-space constrained systems see Bergmann and Goldberg (1955), Bergmann et al. (1956), and Sudarshan and Mukunda (1974).

4. *Nambu Theory*

In the Nambu formalism there seems to be no place for a standard-type definition of canonical momenta (Section 1). In contrast, we shall show that the Poincaré momenta are well defined and give a reasonable result, taking into account the features of Nambu theory. But let us first briefly discuss the standard approach.

4.1. *Lagrangians.* Here we summarize some results given in Kálnay and Tascón (1976). In the present subsection, we shall restrict ourselves to *odd*-dimensional Nambu theory. Since the state vector variables are those whose values for all future times are uniquely determined in terms of the initial values, then it is clear from equation (1.4) that the Nambu state vector variables are *any* of the Nambu phase-space coordinates $x_{\alpha i}$. However, in Hamiltonian dynamics the state vector space is the phase space. Therefore, if unphysical auxiliary variables are not introduced in Nambu and/or Hamilton formalisms, it results that the identification of Nambu with Hamilton dynamics is impossible. This is because the dimensions of the state vector spaces do not fit. [To introduce Dirac-like constraints (Dirac, 1950) does not help, as shown by Cohen and Kálnay (1975).] Therefore, we do not have in Nambu theory a Lagrangian from which via equation (1.1) the momenta could be obtained. However, the *Note* "this result does not necessarily remain correct if the Nambu phase space is implemented with an additional number of auxiliary variables" (which, thanks to a private communication by Professor Ruggeri, we included in Section 2 of Cohen and Kálnay, (1975) is relevant to the above discussion also. In fact, when Nambu coordinates are temporarily transformed in auxiliary variables or when they are implemented by auxiliary variables, Lagrangians can be introduced and equation (1.1) used in order to obtain in a familiar fashion the Nambu theory momenta. For example, the Lagrangian

$$L(\mathbf{x}, \dot{\mathbf{x}}) = H_1(\mathbf{x})\,\dot{\mathbf{x}} \cdot \nabla H_2(\mathbf{x}) \tag{4.1.1}$$

(x_i, $i = 1, 2, 3$ being the configuration variables) proposed by Bayen and Flato (1976) for the Nambu triplet ($\mu = 1, n = 3$) and also considered in Mukunda and Sudarshan (1975) leads to a Hamiltonian embedding of the Nambu equation. Via equation (4.1.1) one can define momenta. This procedure is very interesting and has nothing wrong. However, as regards its use to introduce momenta in Nambu theory, the following two remarks are in order.

(i) The auxiliary variables are not physical variables. They carry no physical data because they are not state vector variables. To specify them at a certain time, leave them completely unspecified in any future time. Therefore, at least one part of the phase-space formalism developed in that way deals with a mixture of physical and unphysical entities. This contrast with the procedure we shall introduce in the next subsection.

(ii) Those momenta are highly nonunique. In fact, other embeddings and other Lagrangians than (4.1.1) are possible. We could obviously use, for example, the Lagrangian

$$L'(\mathbf{Q}, \mathbf{x}, \dot{\mathbf{Q}}, \dot{\mathbf{x}}) = \mathbf{Q} \cdot \{\dot{\mathbf{x}} - [\nabla H_1(\mathbf{x})] \times [\nabla H_2(\mathbf{x})]\} \tag{4.1.2}$$

(Here x_i and Q_i, $i = 1, 2, 3$ are the configuration variables.) Both Lagrangians lead to quite different momenta.

4.2. *The Momentum for Nambu Systems.* Let us consider a Nambu system with multiplets using the same notation as in equations (1.2)–(1.7). Then we prove the following theorem.

Theorem 4.2.1. The integral

$$\sum_{\alpha=1}^{\mu} \int_{\sigma} dx_{\alpha 1} dx_{\alpha 2} \cdots dx_{\alpha n} \tag{4.2.1}$$

is conserved in time.[6]

Proof. Put $F = x_{\alpha i}$ in equation (1.4) and compute

$$\sum_{\alpha i} \frac{\partial \dot{x}_{\alpha i}}{\partial x_{\alpha i}}$$

obtaining zero as the result.

Note 4.2.2. Let us select s values of $i = 1, 2, \ldots, n$ and call

$$q_\alpha \overset{\mathrm{df}}{=} (x_{\alpha i_1}, x_{\alpha i_2}, \ldots, x_{\alpha is}), \qquad \alpha = 1, 2, \ldots, \mu \tag{4.2.2}$$

Notice that the selected values of s will be the same for each value of α. Because of Definition 2.2, *q_α is an s-coordinate of configuration space.* Also each *$x_{\alpha i_k}$ is a coordinate of configuration space (see Definition 2.3).* From Theorem 4.2.1, Note 4.2.2, and Definition 2.4 we obtain the following corollary.

Corollary 4.2.3. The set of all $x_{\alpha i_k}$ not included in the right-hand side of equation (4.2.2), i.e.,

$$f^\alpha = (x_{\alpha,1}, \ldots, x_{\alpha,i_1-1}, x_{\alpha,i_1+1}, \ldots, x_{\alpha,i_s-1}, x_{\alpha,i_s+1}, \ldots, x_{\alpha,i_n}) \tag{4.2.3}$$

is an $(n - s)$-Poincaré momentum conjugated to q_α.

In particular we have the following

MAIN RESULTS 4.2.4. *Each Nambu phase-space coordinate $x_{\alpha i}$, considered as a configuration coordinate, has Poincaré conjugated an $(n - 1)$-Poincaré momentum, which is the set*

$$f^\alpha|_{s=1} = (x_{\alpha,1}, \ldots, x_{\alpha,i-1}, x_{\alpha,i+1}, \ldots, x_{\alpha,n}) \tag{4.2.4}$$

[6] This theorem was first shown by Nambu (1973) for the case of one multiplet.

of all the other $x_{\alpha k}$. Likewise, each Nambu phase-space coordinate $x_{\alpha i}$ can also be considered as a momentum coordinate (Definition 2.5) that is Poincaré-conjugated to the $(n-1)$-configuration-space coordinate by the right-hand side of equation (4.2.4).

Remark 4.2.5. Equations (4.2.3) and (4.2.4) remain correct if we substitute $x_{\alpha i_k} \to c_{i_k} x_{\alpha i_k}$, where the c_{i_k} are arbitrary nonzero constants. This transformation corresponds to $f^\alpha = p^\alpha \to f^\alpha = cp^\alpha$ in the Hamiltonian case (cf. Note 3.4).

Remark 4.2.6. It may be at first surprising that the conjugate to a single coordinate is a set of $n-1$ variables. *But this is just what could be expected from a basic fact of Nambu theory*:

In standard Hamiltonian theory, the time (changed sign) has as conjugate the Hamiltonian. *In Nambu theory there are $n-1$ Hamiltonians* (1.3). Therefore, if Nambu theory shares the main formal properties of the Hamiltonian theory (as expected, see Section 1), *then the number of variables whose set is the generalized momentum conjugated to the time must be just $n-1$.* Therefore, it is reasonable that the number of variables whose set is the generalized momentum conjugated to an $x_{\alpha i}$ also be $n-1$. Moreover, the canonical momenta of standard Hamiltonian theory are generators of translations. Here, we have the following theorem.

Theorem 4.2.7. The generator of translations of a variable $x_{\alpha i}$ in Nambu theory [and in the frame of the Nambu Bracket (1.5) formulation] is, up to a sign, the $(n-1)$-Poincaré momentum (4.2.4) conjugated to $x_{\alpha i}$.

Proof. We must show that

$$\partial g/\partial x_{\alpha i} = \pm \{g, x_{\alpha i}, \ldots, x_{\alpha, i-1}, x_{\alpha, i+1}, \ldots, x_{\alpha, n}\}, \qquad \forall g \quad (4.2.5)$$

which directly follows from equation (1.5).[7] □

[7] Instead of introducing the notion of Poincaré momenta in order to have a generalization of momenta suitable for the Nambu case, a valid procedure could be applied to define the momenta as the generators of translations, i.e., considering 4.2.7 as a definition. However, in this stage of the research of the theory we have not done as just indicated, because (as we shall show immediately) this alternative definition would strongly depend on the role of the different brackets in terms of which Nambu theory can be formulated. In fact, Nambu has shown an n-linear bracket (1.5) as a suitable one to state the theory in the bracket formalism. On the other hand, Ruggeri (1975) has shown that at least for the $\mu = 1$ case, the Nambu theory can also be formulated in a bilinear bracket formalism, with a bracket of the form

$$\{F, G\}^\Gamma = \Gamma_{\alpha\beta} \left(\frac{\partial F}{\partial x_\alpha}\right)\left(\frac{\partial G}{\partial x_\beta}\right) \quad (A)$$

the matrix Γ being singular. In the bilinear bracket formulation of Nambu theory, a definition of momenta as generators of translations would, up to a nonzero constant (cf. Remark 4.2.5), require that

$$\frac{\partial g}{\partial x_\alpha} = \{g, f'^\alpha\}^\Gamma, \qquad \forall g \quad (B)$$

Remark 4.2.8. The sign ambiguity is irrelevant, because of the arbitrariness of sign of the constants c_{i_k} mentioned in Remark 4.2.5. These sign ambiguities also arise in the Hamiltonian case, because of the arbitrariness of sign in the constant c mentioned in Note 3.4.

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f'^{α} being the (eventually different from f^{α}) momentum conjugated to x_α. Equation (B) implies, up to a nonzero constant, that

$$\Gamma_{\alpha\beta}\left(\frac{\partial f'^{\gamma}}{\partial x_\beta}\right) = \delta_{\alpha\gamma} \tag{C}$$

which shows Γ being nonsingular, which is not possible in Nambu theory (Ruggeri, 1975). Therefore, in the bilinear bracket formulation of Nambu theory the coordinates have no conjugated momenta. Then, if generation of translations is used for defining momenta, the result critically depends on which bracket is the suitable bracket. It is not presently clear which of the brackets (1.5) or (A) is more fundamental in Nambu theory. This is an additional reason why we must have a definition (like the Poincaré definition) of momenta, which would be independent of the brackets to be used.

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